

# Living in Curved Momentum Space

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In this paper we review some aspects of relativistic particles' mechanics in the case of a non-trivial geometry of momentum space. We start with showing how the curved momentum space arises in the theory of gravity in 2+1 dimensions coupled to particles, when (topological) degrees of freedom of gravity are solved for. We argue that there might exist a similar topological phase of quantum gravity in 3+1 dimensions. Then we characterize the main properties of the theory of interacting particles with curved momentum space and the symmetries of the action. We discuss the spacetime picture and the emergence of the principle of relative locality, according to which locality of events is not absolute but becomes observer dependent, in the controllable, relativistic way. We conclude with the detailed review of the most studied  $\kappa$ -Poincaré framework, which corresponds to the de Sitter momentum space.

## INTRODUCTION

Quantum gravity is traditionally considered to be a Holy Grail of the modern high energy physics. This theory, when constructed, would provide a missing link between gravity and quantum, which is necessary to complete the 'unfinished revolution'[1] of XXth century physics. Unfortunately, the Quantum Gravity research programme faces not only the well known tremendous technical and conceptual difficulties, but also the seemingly complete lack of experimental feedbacks.

It has been about 15 years ago, when the quantum gravity phenomenology programme was launched[2]. It was noticed that in spite of the fact that the direct observation of Quantum Gravity effects, like scattering of elementary particles at Planckian energies, are well beyond the reach of any foreseeable technology, there may still exist phenomena of quantum gravity origin that might be observable as a result of the presence of powerful amplifiers. For example the minute effect of an interaction of freely moving particles with the quantum space-time foam may get amplified to an observable size if the time of flight of the particle is long enough.

It has been rather clear from the very first days of the quantum gravity phenomenology research programme that the most promising class of effects to look for are those associated with possible modifications of spacetime symmetries of special relativity. Indeed, the flat Minkowski space (or spaces that are flat to a good approximation at the scales of interest) is a configuration of gravitational field with the ten parameter Poincaré group of global spacetime symmetries. It is feasible that there might be effects of quantum gravitational origin that do not vanish in the case when the coarse grained spacetime is Minkowski. But then it is rather natural to expect that these effects may alter the symmetries of Minkowski spacetime, embodied in the Poincaré symmetry group, somehow.

These were the motivations that served as a launching pad for two major research projects: one positing abandoning the Lorentz invariance, leading to construction of models with Lorentz Invariance Violation (LIV) (see[3–5] for a review); and the second, known under the name of Doubly (or Deformed) Special Relativity (DSR), in which much milder alternation of the Poincaré symmetry was posed. In both cases it was implicitly understood that deviations from the standard symmetries of special relativity are of the quantum gravity origin, although the explicit relation relations between the two was never understood in a satisfactory way.

Doubly Special Relativity[6, 7] was based on the intuition that since the diffeomorphism and local Lorentz invariances play such a central role both in both classical and quantum gravity, the Minkowski space should still possess a ten-parameter group of spacetime symmetries and the relativity principle should hold even if quantum gravity effects are taken into account. These effects (or some remnants of them) exhibit themselves through the presence of an additional observer-independent scale of dimension of mass (or length), which can be identified with the Planck mass  $M_{Pl}$  (or Planck length  $l_{pl}$ ), which *deforms* the Poincaré symmetry algebra. Soon after the original papers[6, 7] appeared some explicit models of DSR have been proposed[8–11]. The reader can find more information about DSR in the reviews.[12–14]

Although DSR was devised to be a rather general scheme, it was the so-called  $\kappa$ -Poincaré algebra[15–18] that served as a major example and attracted a lot of attention from the very beginning. This is a Hopf algebra with

dimensionful deformation parameter  $\kappa$  of dimension of mass, which is expected to be of order of Planck mass<sup>1</sup>. We will discuss the  $\kappa$ -Poincaré algebra in more details in Sect. 5 below, but let us just mention its most important feature, namely that contrary to the standard Poincaré algebra of special relativity, in the case of the  $\kappa$ -Poincaré algebra the action of both translations and Lorentz transformation depend on momentum (and spin) of the state it acts on. For example the worldlines of two particles with different momenta are translated, according to  $\kappa$ -Poincaré, by a different, momentum-dependant amounts, which means that the two worldlines may cross for a local observer but miss each other for a translated one.

This leads to an apparent conflict with the locality principle, noticed first in [19] and further discussed in [20, 21]. In the papers [22, 23] the lack of absolute locality was elevated to the status of a new principle, called *The principle of Relative Locality*, which states that when quantum gravity effects are taken into account locality loses its absolute status and becomes relative. It was the major result of these papers to relate relative locality with curvature of momentum space.

The idea that momentum space might be curved is quite old. It seems that it was first spelled out in the paper [24], where it is argued that some kind of ‘reciprocity principle’ should be adopted, stating that both curved spacetime and curved momentum space should be involved simultaneously in the description of (quantum) physics. About ten years later in the seminal paper Snyder argued that curvature in momentum space might be necessary to handle ultraviolet divergencies of quantum field theory [25]. This paper introduces, as a bi-product, a non-commutativity of spacetime coordinates and the minimal length, arguing that both do not need to be in conflict with Lorentz symmetry (for a recent review of the minimal length scenarios see Ref. [26].) The ideas of Snyder was later expanded by the Russian group (see Ref. [27] and references therein.) In the context of DSR and  $\kappa$ -Poincaré algebra it was observed in the papers [28, 29] that both can be naturally understood in terms of the momentum space being a group manifold of the 4-dimensional group  $AN(3)$ , which as a manifold is a submanifold of 4-dimensional de Sitter space. We will return to this construction in Sect. 5.

In all these attempts the introduction of curved momentum space had purely utilitarian character, it was aimed at solving some outstanding problems or to provide a novel technical perspective. The question arises however is there any fundamental reason to believe that the momentum space is actually curved? Although the complete story is not known one can give an argument, ultimately relating curved momentum space with the theory of quantum gravity.

The argument is based on the intuition that the presence of a scale is a prerequisite for emergence of a nontrivial manifold. This intuition was beautifully expressed by Carl Friedrich Gauss already at the dawn of differential geometry:

“The assumption that the sum of the three angles [of a triangle] is smaller than  $180^\circ$  leads to a geometry which is quite different from our (Euclidean) geometry, but which is in itself completely consistent. I have satisfactorily constructed this geometry for myself [...], except for the determination of one constant, which cannot be ascertained a priori. [...] Hence I have sometimes in jest expressed the wish that Euclidean geometry is not true. For then we would have an absolute a priori unit of measurement.”<sup>2</sup>

The necessity of the presence of the scale is easy to understand. Indeed any nontrivial geometry requires nonlinear structures and those can be constructed only if there is a scale that makes it possible to construct nonlinear expressions from fundamental, dimensionful basic variables. One can interpret the Gauss’ dictum as the statement that if a scale of some physical quantity is present in a theory, one could expect that the geometry of the corresponding manifold must be nontrivial. Or putting it in other words: “everything is curved unless it cannot be.”

There are several examples that support this claim. Special relativity introduces a scale of velocity, and according to the Gauss’ dictum one suspects that the manifold of (three) velocities could possess nontrivial structures. And indeed it does. Contrary to Galilean mechanics, in special relativity the velocity composition law is highly nontrivial

$$\vec{v} \oplus \vec{u} = \frac{1}{1 + \vec{u}\vec{v}/c^2} \left( \vec{v} + \frac{\vec{u}}{\gamma_v} + \frac{1}{c^2} \frac{\gamma_v}{1 + \gamma_v} (\vec{v}\vec{u})\vec{v} \right), \quad \gamma_v = \sqrt{1 - \vec{v}^2/c^2}. \quad (1)$$

This expression is neither symmetric nor associative. It is related to deep mathematics [31] and has interesting physical consequences (Thomas precession).

<sup>1</sup> Since the value of the deformation parameter is to be derived from some fundamental theory and/or experiments, in what follows we will use  $\kappa$  to denote the deformation parameter, whose value is not fixed, while the term ‘Planck mass’ will refer to  $M_{Pl} = \sqrt{\hbar/G} \sim 10^{19}$  GeV.

<sup>2</sup> As cited in [30].

The relativistic, four-momentum space is, arguably, even more important physically than the spacetime. Indeed virtually all physical measurements can be reduced to the measurements of energies and momenta of incoming particles of various kinds (probes) performed by measuring devices located at the origin of a coordinate system, and therefore are the momentum space, not spacetime, measurements. It is only by observing the incoming probes that we can infer the properties of distance events [22, 23]. Therefore, the momentum space measurements are physically fundamental and the spacetime properties are inferred from them. (*I don't see space ... I see [images of] things* the renowned Mexican painter Diego Rivera used to say.) The question arises as to if we have good reasons to believe that the momentum space is an almost structureless Minkowski space, or it is conceivable perhaps that it could possess more intricate geometrical structures?

Following the Gauss' intuition a possible way of addressing this question is to look for a theory that could provide us with a momentum scale  $\kappa$ . Such a theory indeed exists. In  $2+1$  spacetime dimensions the Newton's constant  $G$  has the dimension of inverse mass raising the hope that it may provide the sought momentum scale being a prerequisite for the emergence of a nontrivial momentum space geometry. This hope was fully confirmed by the dynamical model calculations and this example will be discussed in some details in Sect. 2.

And what about gravity in the physical  $3+1$  dimensions? Now the Newton's constant is the ratio of Planck length  $l_{Pl}$  and the Planck mass  $M_{Pl}$

$$l_{Pl} = \sqrt{\hbar G}, \quad M_{Pl} = \sqrt{\frac{\hbar}{G}}, \quad (2)$$

and therefore has the dimension of length over mass. However, one can imagine a regime of quantum gravity, in which the Planck length is negligible, while the Planck mass remains finite. This formally means that both  $\hbar$  and  $G$  go to zero, so that both quantum and local gravitational effects become negligible, while their ratio remains finite [22, 23, 32]. In more physical terms this regime is realized if the characteristic length scales relevant for the processes of interest are much larger than  $l_{Pl}$ , so that the spacetime quantum foamy effects can be safely neglected, while the characteristic energies are comparable with the Planck energy. An example of such a process might be the gravitational scattering in the case when the longitudinal momenta are Planckian, while the transferred momentum is very small (as compared to  $M_{Pl}$ ) [33], [34]. In the case of such processes we again encounter the situation that the momentum scale is present, and we expect to find a nontrivial geometry of the momentum space. Unfortunately, to date no specific model of this kind has been formulated.

The plan of this review is as follows. In the next section we will present a discussion of  $2+1$  gravity coupled to particle(s) and we will see how curved momentum space arises in this model. We will also briefly comment on a similar construction that might be possible in the context of  $3+1$  gravity. In Sect. 3 we will present a model of particles dynamics in the case of an arbitrary geometry of momentum space. Sect. 4 is devoted to the discussion of spacetime structures emerging from this construction and to relative locality. Section 5 will present description of  $\kappa$ -Poincaré momentum space geometry and the particles' model with this momentum space.

## CURVED MOMENTUM SPACE FROM GRAVITY

Before turning to the discussion of the most general curved momentum space let us consider an explicit model, which will provide an intuition as to how gravity could lead to a nontrivial geometry of momentum space.

We start with the gravity in  $2+1$  dimensions. In this case the gravitational lagrangian has the dimension of inverse length square and thus the Newton's constant  $G$  has the dimension of inverse mass  $4\pi G = \kappa^{-1}$ , where  $\kappa$  is the  $2+1$  dimensional Planck mass. According to the argument presented in the Introduction, we can expect that when particles become coupled to gravity their effective momentum space will possess a nontrivial geometry, with the characteristic scale  $\kappa$ . Let us see explicitly how this comes about.

It has been shown by Witten [35, 36] that gravity in  $2+1$  spacetime dimensions can be formulated as a Chern-Simons topological field theory with the gauge group being (in the case of vanishing cosmological constant) the  $2+1$  dimensional Poincaré group  $ISO(2, 1)$ , whose generators satisfy the following commutational relations

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, T_b] = \epsilon_{abc} T^c, \quad [T_a, T_b] = 0.$$

In what follows we will call  $J_a$  and  $T_a$  Lorentz and translational generators, respectively. In terms of these generators the connection one-form decomposes into spin connection  $\omega^a$  and dreibein  $e^a$

$$A = \omega^a J_a + e^a T_a \quad (3)$$

and the action takes the form

$$S = \frac{1}{4\pi G} \int \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle, \quad (4)$$

where  $\langle \star \rangle$  denotes an invariant inner product on the Poincaré algebra defined by[36]

$$\langle J_a T_b \rangle = \eta_{ab}, \quad \langle T_a T_b \rangle = \langle J_a J_b \rangle = 0. \quad (5)$$

One can show by direct calculation[36] that this action with the connection given by (3) reduces to the standard action for gravity in 2+1 dimensions.

It will be convenient to what follows to assume that the 2+1 manifold  $\mathcal{M}$  has the product structure of time times space  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  and to decompose the connection  $A$  accordingly

$$A = A_0 dt + A_s, \quad A_s = A_i dx^i, \quad i = 1, 2 \quad (6)$$

Let us now turn to the coupling of 2+1 gravity to a point particle[37]. One can use fix the diffeomorphism invariance on the particle's wordline in such a way so as to put the particle into rest at the origin  $\mathbf{x} = \mathbf{0}$ . In this case the particle is characterized by its mass  $m$  and spin  $s$ , both being the charges associated with the group of spacetime symmetries and therefore can be written collectively as an element of the Cartan subalgebra of the gauge algebra  $Q = mJ_0 + sP_0$ . In the following, for simplicity, we will assume that the particle's spin vanishes,  $s = 0$ .

The particle at rest is fully characterized by its wordline and the Lie algebra valued charge  $Q$ , and the simplest, minimal coupling to gravity takes the form

$$S_{int}^{(0)} = \int d^3x \langle A_0 Q \rangle \delta^2(\mathbf{x}) \quad (7)$$

The action (7) is not only manifestly diffeomorphism not-invariant, it also breaks the gauge symmetry on the particle's wordline. This is actually a desired feature of the formalism, because, as we will see, the gauge degrees of freedom along the wordline become the dynamical degrees of freedom of the particle. Since the away from the particle the theory is topological and degrees of freedom of gravity are pure gauge, the degrees of freedom along the wordline are the only 'real' degrees of freedom in the theory (if the space manifold  $\mathcal{S}$  has simple enough topology.)

Indeed plugging the gauge transformed field

$$A_0^{\mathfrak{h}} = \mathfrak{h}^{-1} A_0 \mathfrak{h} + \mathfrak{h}^{-1} \dot{\mathfrak{h}} \quad (8)$$

we get

$$S_{int} = \int d^3x \langle A_0 \mathfrak{h} Q \mathfrak{h}^{-1} \rangle \delta^2(\mathbf{x}) + \int dt \langle \mathfrak{h}^{-1} \dot{\mathfrak{h}} Q \rangle. \quad (9)$$

The first term describes the coupling of the particle with the gravitational field, while the second is the particle's kinetic term. To see this let us decompose the Poincaré group element  $\mathfrak{h}$  into Lorentz and translational parts  $\mathfrak{h} = (\mathbf{u}, \mathbf{r})$ ,  $\mathfrak{q} = q^a T_a$  in terms of which the second term in (9) takes the form

$$\int dt m \langle \dot{q}^a T_a \mathbf{u} J_0 \mathbf{u}^{-1} \rangle = \int dt \dot{q}^a p_a. \quad (10)$$

where we define the momenta  $p_a$  by the formula  $m \mathbf{u} J_0 \mathbf{u}^{-1} \equiv p_a J^a$ . This procedure has clear physical interpretation. A Poincaré group element describe Lorentz transformation and translation. We use the Lorentz part to boost the particle from its rest state to the actual state of motion characterized by momentum  $p_a$ ; the translation moves the particle's position from the origin to its actual position  $q^a$ .

As mentioned already gravity in 2+1 dimensions is described by a topological field theory, and therefore it does not posses local degrees of freedom: neither Newtonian attraction nor gravitational waves are possible. This clearly follows from the fact that the field equations following from (4) force both the curvature of  $\omega$  and torsion to vanish and therefore the spacetime of 2+1 gravity, locally at least, is flat. In the case of the particles coupling the situation is similar: the field equations says that the curvature and torsion are zero everywhere except at the very position of the particle, where they acquire a delta-like singularity[38]. This does not mean however that all the dynamics is gone. The particle's dynamical degrees of freedom are still there. The presence of gravity does not change the number of

degrees of freedom in the theory. It does however a remarkable thing: gravitational degrees of freedom are absorbed by the particle's ones effectively deforming its the kinematics and dynamics.

Let us sketch how this comes about (we refer the reader to the upcoming paper[39] for more details and detailed discussion of the subtle points of the construction). We start with the kinetic term of the action (4) written down with the help of the decomposition (6)

$$S_{kin} = \frac{1}{4\pi G} \int dt \int_S \langle A_s \wedge \dot{A}_s \rangle + \langle m J_0, \mathfrak{h}^{-1} \dot{\mathfrak{h}} \rangle \delta^{(2)}(\mathbf{x}), \quad (11)$$

where the connection one-form  $A_s$  is constrained by the Gauss law (the field equation of  $A_0$ )

$$\frac{1}{2\pi G} F(A_s) = m \mathfrak{h} J_0 \mathfrak{h}^{-1} \delta^{(2)}(\mathbf{x}) dx^1 \wedge dx^2. \quad (12)$$

The constraint (12) can be easily solved as follows. Let us decompose the 2-dimensional space manifold  $\mathcal{S}$  into two subregions: the plaquette  $\mathcal{D}$  being a circle with the center at the position of the particle, on which we introduce coordinates  $0 \leq r \leq 1$  and  $0 \leq \phi \leq 2\pi$  and the asymptotic region  $\Sigma$  with  $r \geq 1$ . These two regions have a common boundary  $\mathcal{H}$ ,  $r = 1$ ,  $0 \leq \phi \leq 2\pi$ . On the asymptotic region the connection is flat and the gauge field takes the form

$$A_\Sigma = \mathfrak{g}^{-1} d\mathfrak{g} \quad (13)$$

where  $\mathfrak{g}$  is an element of the Poincaré gauge group. One can find a general solution of (12) on the disc as well, it reads<sup>3</sup>

$$A_{\mathcal{D}} = G m \bar{\mathfrak{g}}^{-1} J_0 \bar{\mathfrak{g}} d\phi + \bar{\mathfrak{g}}^{-1} d\bar{\mathfrak{g}}, \quad \mathfrak{g}(0) = \mathfrak{h}^{-1}. \quad (14)$$

The fact that (14) is a general solution of (12) on the disk can be easily checked using the identity  $dd\phi = 2\pi\delta(\mathbf{x})dx^1 \wedge dx^2$ . In addition we assume that the gauge field is continuous across the boundary  $\mathcal{H}$ , which imposes an additional relation between  $\mathfrak{g}$  and  $\bar{\mathfrak{g}}$

$$\mathfrak{g}^{-1} d\mathfrak{g}|_{\mathcal{H}} = G m \bar{\mathfrak{g}}^{-1} J_0 \bar{\mathfrak{g}} d\phi + \bar{\mathfrak{g}}^{-1} d\bar{\mathfrak{g}}|_{\mathcal{H}}. \quad (15)$$

Decomposing  $\mathfrak{g} = (\mathbf{u}, \mathbf{q})$ ,  $\bar{\mathfrak{g}} = (\bar{\mathbf{u}}, \bar{\mathbf{q}})$  as before, a general solution of (15) can be found to read[40]

$$\mathbf{u} = \mathbf{n} e^{Gm J_0 \phi} \bar{\mathbf{u}}, \quad \mathbf{q} = \eta + Ad(\mathbf{n} e^{Gm J_0 \phi}) \bar{\mathbf{q}}. \quad (16)$$

where  $\mathbf{n}$  and  $\eta$  are time-dependent (but space-independent) Lorentz and translation groups elements, respectively. Notice that in the transition from (14) to (15) the Lie algebra valued momentum of a free particle at rest  $m J_0$  (cf. (8) and discussion that follows) becomes replaced by group valued one  $e^{Gm J_0 \phi}$ , when gravitational gauge degrees of freedom are taken into account.

A remarkable thing happens when the explicit forms of the Chern–Simons connection (13), (14) with the boundary conditions (16) are substituted to the action (11). Namely, the action of the gravity plus particle system collapses to the action describing a deformed particle with curved momentum space being an  $SO(2, 1)$  group manifold. Explicitly, in terms of the group valued momenta

$$\Pi \equiv \mathbf{u} e^{2\pi Gm J_0} \mathbf{u}^{-1} = p_3 \mathbf{1} + \frac{p_a}{\kappa} J^a, \quad (17)$$

where the momenta are coordinates on the group manifold of  $SO(2, 1)$ , which geometrically is a 2+1 dimensional anti de Sitter space

$$p_3^2 - p_a p^a = \kappa^2, \quad \kappa = (2\pi G)^{-1} \quad (18)$$

subject to the mass shell condition<sup>4</sup>

$$p_a p^a = -m^2. \quad (19)$$

<sup>3</sup> This follows immediately from the fact that  $F(A^{\mathfrak{g}}) = \mathfrak{g}^{-1} F(A) \mathfrak{g}$ , where  $A^{\mathfrak{g}}$  denotes the gauge transformed connection and  $F(Gm J_0 d\phi) = Gm J_0 dd\phi = 2\pi Gm J_0 \delta(\mathbf{x}) dx^1 \wedge dx^2$ .

<sup>4</sup> In this paper we use the signature  $(-, +, +)$  in 2+1 dimensions and  $(-, +, +, +)$  in 3+1 dimensions.

This effective kinetic term reads

$$S_{kin} = \int dt \left\langle \Pi^{-1} \dot{\Pi} T_a \right\rangle x^a + N(p^2 + m^2), \quad (20)$$

where is a standard Lagrange multiplier enforcing the mass shell constraint. Notice that the kinetic term is written in terms of the Kirillov symplectic form[65], which is a natural symplectic form on a group manifold. In components the action (20) this effective kinetic term reads

$$S_{kin} = - \int dt p_3 \dot{p}_a x^a + \frac{1}{\kappa} \epsilon^{abc} \dot{p}_a x_b p_c - \frac{1}{\kappa^2} \dot{p}_a p^a p_b x^b + N(p^2 + m^2), \quad p_3^2 = 1 + \frac{1}{\kappa^2} p_a p^a \quad (21)$$

Notice that in the limit  $\kappa \rightarrow \infty$  (i.e., in the no-gravity limit  $G \rightarrow 0$ ) this action reproduces the free relativistic particle action (10).

Anticipating the discussion of a more general case to be presented in the next section, let us notice that the action (21) can be neatly written down as

$$S_{kin} = - \int dt \dot{p}_\alpha E^\alpha{}_b(p) x^a + N(p^2 + m^2), \quad (22)$$

where  $E^\alpha{}_b(p)$  is a dreibein on the momentum space, and to stress that the momentum space is curved we used the Greek index  $\alpha$  to label components of the momentum.

It is not completely obvious if the construction presented above can be extended without modifications to the case of many particles, but this happen to be true nevertheless[40]: for the finite number of particles the kinetic term turns out to be a sum of terms of the form (20). This construction can be extended further to the case of a (not self interacting) scalar field coupled to gravity[41, 42].

Till now we discussed only free particle(s). Let us now consider a bunch of particles, which, in addition to their coupling to the gravitational field considered above have some other interactions of not topological nature. Assume that these interactions are contact in a sense that there is a well localized vertex in which the interaction takes place and one can think of asymptotic regions where the interactions can be neglected. Let us consider the simplest possible, nontrivial vertex in which two incoming particles interact and form a single outgoing particle. The law of momentum conservation at the vertex is the postulate that the total momentum of the initial two particle configuration equals the momentum of the outgoing one. Therefore the problem of finding the momentum conservation rule reduces to the problem to find out what is the correct measure of total momentum of the two-particles initial state. Since, as we have seen, the momenta become coordinates on a group manifold it is natural to expect that the total momentum is given by the product of the corresponding group elements[43]

$$\exp \left( p_a^{(tot)} J^a \right) = \exp \left( p_a^{(1)} J^a \right) \exp \left( p_a^{(2)} J^a \right), \quad (23)$$

which can be generalized to the case of an arbitrary group momentum space. Introducing the operation  $\oplus$  such that

$$p_a^{(tot)} = p_a^{(1)} \oplus p_a^{(2)} \quad (24)$$

we define it by demanding that

$$g \left( p^{(1)} \oplus p^{(2)} \right) = g \left( p^{(1)} \right) g \left( p^{(2)} \right). \quad (25)$$

The case of 3+1 gravity is by far less understood but there are some partial results that make it reasonable to claim that there may exist a regime of this theory quite similar to the 2+1 dimensional case discussed above.

In 3+1 dimensions gravity is certainly *not* described by a topological field theory, as it was the case in 2+1 dimensions, but it turns out that it is surprisingly ‘close’ to such a theory. In fact, the 3+1 gravity can be described by a ‘constrained’ BF theory, which means that Einstein lagrangian can be written down as a sum of the topological BF theory lagrangian and a small symmetry breaking term (for a recent review and extensive references to earlier works see Ref. [44].)

Specifically, one starts the construction introducing the connection of de Sitter gauge group

$$A = \omega^{ij} T_{ij} + \frac{1}{\ell} e^i T_{i4}, \quad (26)$$

where  $T_{IJ} = (T_{ij}, T_{i4})$ ,  $I, J = 0, \dots, 4$ ,  $i, j = 0, \dots, 3$  are dimensionless generators of the  $SO(4, 1)$  (de Sitter) group and  $\ell$  is a length scale needed for the dimensional reasons, because we want the tetrad to be dimensionless<sup>5</sup>. We also introduce a  $so(4, 1)$  Lie algebra valued two form field  $B = B^{AB} T_{AB}$ . In terms of these fields the action reads[45, 46]

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} B^{IJ} \wedge F_{IJ}(A) - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{\alpha}{4} \epsilon_{ijkl} B^{ij} \wedge B^{kl}. \quad (27)$$

The first two terms (for any value of the parameter  $\beta$ ) form the action of the topological BF theory. The last term manifestly breaks the gauge symmetry down to the Lorentz group  $SO(3, 1)$  and is essential for the emergence of local degrees of freedom of gravity. Because of the presence of this last term the theory described by (27) is called the ‘constrained BF theory.’ It is worth noticing that when the limit  $\alpha \rightarrow 0$  is taken (27) to the action of a pure topological theory with no dynamical degrees of freedom.

It comes as a pleasant surprise that the theory described by (27) is in fact equivalent to the Einstein-Cartan theory. To see this one has to solve the algebraic equations for  $B$  field and substitute the result back to the action. As a result one gets the Holst action[47]

$$32\pi G S = \int R^{ij} \wedge e^k \wedge e^l \epsilon_{ijkl} + \frac{\Lambda}{6} \int e^i \wedge e^j \wedge e^k \wedge e^l \epsilon_{ijkl} + \frac{2}{\gamma} \int R^{ij} \wedge e_i \wedge e_j. \quad (28)$$

supplemented by topological invariants: Euler, Pontryagin, and Nieh-Yan classes[46]. The physical coupling constants, the Newton’s constant  $G$  and the Immirzi-Barbero parameter  $\gamma$  are related to the coupling constants of the original action (27) as follows

$$G = \frac{\alpha^2 + \beta^2}{\alpha} \frac{1}{\Lambda}, \quad \gamma = \frac{\beta}{\alpha}, \quad \Lambda = \frac{3}{\ell^2} \quad (29)$$

It can be shown that the constrained BF theory described by (27) can be coupled to point particles with momentum and spin exactly as it was done for 2+1 gravity above[48].

In the introductory section we argued that the curved momentum space may arise if one takes the limit of quantum gravity  $\ell_{Pl} \rightarrow 0$  with  $M_{Pl}$  kept fixed, or putting it in another way,  $G \rightarrow 0$ ,  $\hbar \rightarrow 0$  with their ratio fixed. Having the theory (27) coupled to particles one can look for curved momentum space in the effective theory arising in the topological limit  $\alpha \rightarrow 0$ , which reminds in many respects the 2+1 dimensional situation. Notice that according to (29) this is again the limit of zero gravitational and Planck constants, so it is possible that the theory operating in this limit may describe a curved momentum space, at least in some circumstances. Some more quantitative works based on this general intuition have been done in the past[49, 50], but the results were not conclusive.

In the rest of this review we will assume that there are physical systems that require the curved momentum space for their description and we will present the techniques necessary to analyze properties of such systems.

## RELATIVISTIC PARTICLES WITH CURVED MOMENTUM SPACE

In the previous section we saw how in 2+1 spacetime dimensions the free relativistic particle action becomes deformed by absorbing topological degrees of freedom of gravity. The net result of this absorption process is that the momentum space of the particle becomes a curved manifold (a group manifold) and the components of the particle’s momentum become (coordinate) functions on this manifold. In the case of the physical 3+1 spacetime dimensions the analogous derivation is still missing (although, as we saw, there are some arguments that it may work as well) and therefore here we will assume that the geometry of momentum space is arbitrary. This will make it possible to identify the geometric object that are necessary to make the transition from the standard formulation of the theory of relativistic particles to the one appropriate to the case of a nontrivial momentum space geometry.

To see how this generalization can be implemented let us start with the discussion of the action of a free relativistic particle. It reads

$$S^0 = - \int_{-\infty}^{+\infty} d\tau x^a \dot{p}_a + N (\eta^{ab} p_a p_b + m^2), \quad (30)$$

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<sup>5</sup> The canonical dimension of the gauge field potential is inverse length, thus the connection one form is dimensionless.

where the overdot denotes differentiation with respect to the parameter  $\tau$ . The lagrangian in (30) consists of two terms: the kinetic one  $-x^a \dot{p}_a$ ,  $a, b = 0, \dots, 3$  and the mass shell constraint  $\eta^{ab} p_a p_b + m^2$  imposed by the Lagrange multiplier  $N(\tau)$ . It will be important for the later purposes to note that the term  $\eta^{ab} p_a p_b$  is nothing but the square of the Minkowski distance between the point  $\mathcal{P}$  in momentum space, with coordinates  $p_a$  and the momentum space origin  $\mathcal{O}$  with coordinates  $p_a = 0$ , calculated along the straight line joining these two points, i.e., along the geodesic of the Minkowski space geometry.

Let us note in passing that the action (30) is manifestly invariant under global Lorentz and local  $\tau$  reparametrization symmetries, as well as global translations

$$\delta_\xi x^a = \xi^a, \quad \delta_\xi p_a = \delta_\xi N = 0. \quad (31)$$

The equations of motion resulting from (30) are

$$\dot{p}_a = 0, \quad \dot{x}^a = 2N \eta^{ab} p_b, \quad \eta^{ab} p_a p_b = -m^2. \quad (32)$$

The first equation is the momentum conservation, the second relates velocity to momentum, while the third is the mass-shell condition.

The system of several non-interacting relativistic particles labeled by  $\mathcal{I}$  is described by the action being a sum of the actions (30),

$$S_{free}^0 = - \sum_{\mathcal{I}} \int d\tau x_{\mathcal{I}}^a \dot{p}_a^{\mathcal{I}} + N_{\mathcal{I}} (\eta^{ab} p_a^{\mathcal{I}} p_b^{\mathcal{I}} + m_{\mathcal{I}}^2). \quad (33)$$

We can introduce particles interactions as follows. Let some number of wordlines meet at the interaction vertex, and let us assume that at the vertex the momenta are conserved. One can further assume that for the wordlines corresponding to incoming particles the parameter  $\tau$  has the range from  $-\infty$  to 0, while for the outgoing ones from 0 to  $\infty$ , and the interaction point corresponds to the  $\tau = 0$  on each wordline, but from now on we will not write the range explicitly.

To include the interaction one adds to the action (33) an interaction term of the form[22]

$$S_{int}^0 = z^a \mathcal{K}_a(p^{\mathcal{I}}), \quad \mathcal{K}_a(p^{\mathcal{I}}) \equiv \widetilde{\sum_{\mathcal{I}} p_a^{\mathcal{I}}} \quad (34)$$

where the tilde over sum indicates that the incoming momenta are taken with plus, while outgoing with minus signs. We assume that the total action is  $S_{tot} = S_{free} + S_{int}$ . Along the wordlines the equations of motion following from this total action do not change and take the form (32) for each particle and variation over  $z^a$  results in the momentum conservation rule at the vertex. However, since the wordlines are semi-infinite now, when varying the free action over momentum, from each particle action we get a boundary term that gets combined with the variation of  $\mathcal{K}_a$  leading to the condition

$$x_{\mathcal{I}}^a(0) = z^a \quad \forall \mathcal{I}. \quad (35)$$

This equation says that the ‘local interaction coordinate’  $z^a$  is equal to the coordinates of the ends of the wordlines  $x_{\mathcal{I}}^a(0)$ . It is worth noticing that eq. (35) is covariant under Lorentz transformations and invariant under translations, if we take  $\delta_\xi Z^a = \xi^a$ . We see therefore that in special relativity locality is *absolute*: if an event (interaction of particles) is local for one inertial observer it is local for other inertial observers, in the sense that for all observers  $x_{\mathcal{I}}^a(0) = x_{\mathcal{J}}^a(0) = z^a$  for all  $\mathcal{I}, \mathcal{J}$ .

It should be noted that the absolute locality exhibited by the relativistic particles model relies on the ‘correct’ choice of the coordinates on the particles’ phase space. Indeed, had we chosen another position coordinates  $x^a \mapsto X^a \equiv M_b^a(p) x^b$  the theory would suffer from an apparent lack of locality: under translation the wordlines would transform in a momentum dependent way, i.e., instead of (31) we would have  $\delta X^a = M_b^a(p) \xi^b$ . We will return to this point in Sect. 4 below while discussing relative locality.

Now we want to generalize this setup to the case of curved momentum space[22]. As we saw in the previous section to define the kinetic term one needs to employ the momentum space tetrad  $E_a^\alpha(p)$ <sup>6</sup>. As for the dispersion relation

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<sup>6</sup> From now on, to stress that momenta are coordinates on a curved manifold we label them with a Greek index.



we noticed above that in the case of special relativity it can be interpreted as a geodesic distance in Minkowski space between the spacial ‘zero momentum’ point  $\mathcal{O}$  with coordinates  $p_\alpha = 0$  and the point  $\mathcal{P}$  with the actual coordinates  $p_\alpha$ . In the case of a curved momentum space manifold this generalizes to a square of the geodesic distance  $\mathcal{C} = D^2(p)$ , calculated with the help of the metric  $g^{\alpha\beta}(p) = E_a^\alpha(p) E_b^\beta(p) \eta^{ab}$ . Taken all this together we get

$$S_{free}^\kappa = - \sum_{\mathcal{I}} \int d\tau x_{\mathcal{I}}^a E_a^\alpha(p^{\mathcal{I}}) \dot{p}_\alpha^{\mathcal{I}} + N_{\mathcal{I}} (\mathcal{C}(p^{\mathcal{I}}) + m_{\mathcal{I}}^2) . \quad (36)$$

Assuming invertibility of the momentum space tetrad  $E_a^\alpha(p^{\mathcal{I}})$  the equations of motion for  $x_{\mathcal{I}}^a$  force the momenta to be constant along the worldlines,  $\dot{p}_\alpha^{\mathcal{I}} = 0$ . Further from the equation for  $N$ , it follows that the deformed mass shell condition  $\mathcal{C}(p) = -m^2$  has to be satisfied. Finally, the equation for  $p_\alpha^{\mathcal{I}}$  imposes the following relation between momenta and velocities

$$\dot{x}_{\mathcal{I}}^a = N E_a^\alpha(p^{\mathcal{I}}) \frac{\partial}{\partial p_\alpha^{\mathcal{I}}} \mathcal{C}(p^{\mathcal{I}}) . \quad (37)$$

Notice that  $N$  can be reabsorbed into redefinition of the parameter  $\tau \rightarrow \bar{\tau}$ ,  $d\bar{\tau}/d\tau = N$ , as usual, so it can be gauge fixed to be equal to an arbitrary positive number.

One can calculate the Poisson brackets on the phase space of the theory described by (36) by simply noticing that the pairs  $(x_{\mathcal{I}}^a(p), p_\beta^{\mathcal{I}})$ , where  $x_{\mathcal{I}}^a(p) = x_{\mathcal{I}}^a E_a^\alpha(p^{\mathcal{I}})$  have the canonical Poisson bracket

$$\{x_{\mathcal{I}}^a, p_\beta^{\mathcal{I}}\} = \delta_{\mathcal{I}}^{\mathcal{J}} \delta_\beta^\alpha ,$$

from which the brackets for  $(x_{\mathcal{I}}^a, p_\beta^{\mathcal{I}})$  can be read off (see (52) and (53) below for explicit expressions.)

Having discussed the deformed free particles’ action let us now turn to interactions. Recalling eq. (34) we see that in order to define the momentum conservation rule at the vertex we must first introduce the operation  $\oplus$  such that the total momentum of two particles having momenta  $p_\alpha$  and  $q_\alpha$  is

$$p_\alpha^{tot} = (p \oplus q)_\alpha . \quad (38)$$

In principle the operation  $\oplus$  does not need to be neither symmetric (i.e.,  $p \oplus q$  could be not equal to  $q \oplus p$ ) nor associative (i.e.,  $(p \oplus q) \oplus r$  could be not equal to  $p \oplus (q \oplus r)$ . ) In the case of the momentum space being a group manifold,  $\oplus$  is constructed from the group composition rule and is associative (because the group composition is) but not symmetric (for non-abelian groups.) We also need the operation of inverse (or antipode in the language of Hopf algebras) denoted  $\ominus$  which generalizes the ‘minus’ and satisfies

$$p \oplus (\ominus p) = (\ominus p) \oplus p = 0 \quad (39)$$

Using the operators  $\oplus$  and  $\ominus$  we can generalize the form of the interaction term, eq. (34) to read

$$S_{int}^\kappa = z^\alpha \mathcal{K}_\alpha(p^{\mathcal{I}}), \quad \mathcal{K}_\alpha(p^{\mathcal{I}}) \equiv \left( \widetilde{\bigoplus_{\mathcal{I}}} p^{\mathcal{I}} \right)_\alpha \quad (40)$$

and the analogue of eq. (35) takes the form

$$x_{\mathcal{I}}^a(0) = E_\beta^a z^\alpha \frac{\partial \mathcal{K}_\alpha}{\partial p_\beta^{\mathcal{I}}} . \quad (41)$$

Let us comment at this point that to define  $\mathcal{K}$  we use the operator  $\tilde{\bigoplus}$  with outgoing particles contributing with the  $\ominus$  to the sum. This corresponds to the physical assumption that the velocities of the particles are timelike, future directed for both incoming and outgoing particles. Alternatively one can use the  $\bigoplus$  operation, with the understanding that for the outgoing particles the velocity is past directed.

Let us finish this part with some comments on the theory of interacting particles with curved momentum space formulated above.

- It should be stressed once again that the construction presented above would not be possible had it not been for the presence of the momentum scale. This is clearly visible if one attempts to expand the functions introduced above in powers of momenta, near the origin  $\mathcal{O}$ ,  $p = 0$ . For momentum space tetrad we have, for example

$$E_a^\alpha(p) = \delta_a^\alpha + \frac{1}{\kappa} C^{\alpha\beta}_a p_\beta + \dots$$

and since  $E_a^\alpha(p)$  is dimensionless by definition  $\kappa$  must have dimension of mass.

- The momentum conservation rule (40) is, in general, neither symmetric, nor associative nonlinear function of the momenta  $p^\mathcal{I}$ . This means that for given momenta of the incoming particles there might be many reaction channels with outgoing momenta being different in each. However this effect will be at least of order  $1/\kappa$  and is certainly not in conflict with any available experimental data. Moreover, the existence of many channels of momentum conservation does not contradict any fundamental physical postulate.
- As discussed in details in [22] and [51] there are interesting and deep geometrical structures behind the momentum composition rule. As shown there any non-trivial momentum composition can be geometrically expressed in terms of a connection on momentum manifold and vice versa, any connection defines a composition rule. This connection is defined (at the origin  $\mathcal{O}$ ; the expression at arbitrary point can be found in [22, 51])

$$\Gamma_\gamma^{\alpha\beta} = - \frac{\partial^2}{\partial p_\alpha \partial q_\beta} (p \oplus q)_\alpha \Big|_{p=q=0}$$

If the composition rule is not symmetric, the connection has torsion

$$T_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta} - \Gamma_\gamma^{\beta\alpha};$$

if it is not associative, it has a non-vanishing curvature

$$R_\gamma^{\alpha\beta\sigma} = \frac{\partial^3}{\partial p_{[\alpha} \partial q_\beta] \partial r_\gamma} [(p \oplus q) \oplus r)_\alpha - (p \oplus (q \oplus r))_\alpha] \Big|_{p=q=r=0}$$

Moreover, the metric introduced in the course of defining the free action (36) does not need to be (and usually is not) connection compatible, i.e., the covariant derivative of the connection applied to the metric does not vanish in general  $\nabla^\alpha g^{\beta\gamma} \equiv N^{\alpha\beta\gamma} \neq 0$ . It turns out that the abstract theory of momentum is closely related to the mathematical theory of loops, see [52] and [53] for more details and references to the original mathematical literature.

It is worth noticing that in the best understood, and physically most important cases, the momentum composition law is a group product, which is, by definition associative and, in general, non-commutative. It follows that in the case of group-like momentum composition law the curvature vanishes, but the torsion is generally not zero. Usually the metric one uses to define the free particle action is not connection-compatible.

- It should be stressed once again that it follows from eq. (41) that contrary to the special relativistic case (34) the value of  $x^a$  at the end of the trajectory depends on the momenta of all particles that interact in the vertex (including its own, in general). This is the signal of a completely new spacetime physics and emergence of relative locality, which we will discuss below.
- Last but not least let us comment on an important conceptual problem that apparently plagues any theory with a nonlinear momentum space deformation. One could argue that such deformations clearly contradict the everyday experience that macroscopic bodies satisfy the standard linear on-shell relations and linear momentum conservation rules. And yet the momenta of macroscopic bodies are by many orders of magnitude larger than the scale  $\kappa$ , and thus for them the nonlinear deformation terms should be particularly large. This problem is dubbed *the soccer ball problem*. To resolve this paradox one notices that macroscopic bodies are composite systems, built from large number  $N$  of elementary constituents. Then it can be argued that, for example, in the macroscopic scattering problem we have to do not with one interaction process, but with a huge number of elementary interactions, between elementary constituents, which as a result re-scale the deformation scale to become of order of  $N\kappa$ . Thus for a soccer ball the deformation scale is at least of order of  $10^{23} M_{Pl} \sim 10^{18} \text{g}$  (see [54] for more details and references.) Let us notice at this point that as another side of the same coin we can ask ourselves if we have good reason to believe that the Standard Model elementary particles are indeed elementary or ‘soccer balls’ of some kind, or, in other words, do we have good reason to believe that the scale  $\kappa$  is indeed of order of the Planck mass. The answer to this question rests with the experiment.

### Symmetries of the action

As we briefly discussed above, the action of a particle in special relativity is invariant under global translational and Lorentz symmetries. Let us now investigate what is the fate of these symmetries in the theory of particles with curved momentum space.

Consider first the translational symmetry. For simplicity let us assume that we have 3 particles meeting at the vertex labeled by the interaction coordinate  $z^\alpha$  and that the range of the parameter  $\tau$  is  $(-\infty, 0)$  for all three wordlines. As in the case of special relativity the translational symmetry leaves the momenta invariant  $\delta_\xi p^\mathcal{I} = \delta_\xi N^\mathcal{I} = 0$ . Therefore the condition for translational invariance takes the form

$$\sum_{\mathcal{I}} \int_{-\infty}^0 d\tau \delta x_{\mathcal{I}}^a E_a^\alpha(p^\mathcal{I}) \dot{p}_\alpha^\mathcal{I} - \delta z^\alpha \mathcal{K}_\alpha(p^\mathcal{I}) = 0.$$

If we now take

$$\delta x_{\mathcal{I}}^a = \xi^\gamma \frac{\partial \mathcal{K}_\gamma}{\partial p_{\mathcal{I}}^a} E_\beta^a \quad (42)$$

the sum of integrands over  $\mathcal{I}$  gives a total derivative  $d(\xi^\gamma \mathcal{K}_\gamma)/d\tau$  and the translational invariance condition reduces to  $\delta z^\alpha = \xi^\alpha$ . The wordline labels  $x_{\mathcal{I}}^a$  transform therefore in a very complex way (42), depending, in general, on the momenta of all the particles that it meets in the vertex, while the interaction coordinate  $z^\alpha$  is becoming translated by a constant, just like in special relativity.

It is considerably harder to prove translational invariance for a ‘tree process’ with many vertices (meaning the particles created at one vertex to not interact again), and there are indications that when loop processes are allowed the translational invariance might be inevitably lost (see Ref. [66] for more details.) In the multi-vertex situation, the translational invariance is in a sense ‘holistic’ since its action on a particular particle wordline carries an information about all other particles in the universe the particle in question has interacted or will interact with.

Let us now turn to Lorentz symmetry. It should be noted that since, as discussed in the preceding section, one uses different and a priori independent geometric objects to construct free action (36) and the interaction term (40) Lorentz invariance is manifestly violated in general. As we will see in a moment it is always possible to make the free action Lorentz invariant; then the interaction term may turn out to be invariant as well, or not (for example one may have a nontrivial metric, and therefore a nontrivial mass-shell relation and the standard linear conservation law.) In what follows we will investigate under which condition the total action is Lorentz invariant.

Let us assume for simplicity that the action of rotational subgroup of the Lorentz group is un-deformed (which means that the momentum space tetrad and the connection transform as tensors of appropriate rank under the action of rotations). Consider the mass-shell relation in the free action first. It is Lorentz invariant if there exist three ‘generators’ (Hamiltonian vector fields)  $N_i$  satisfying the Poisson bracket relations

$$\{M_i, N_j\} = \epsilon_{ij}^{\phantom{ij}k} N_k, \quad \{N_i, N_j\} = -\epsilon_{ij}^{\phantom{ij}k} M_k, \quad (43)$$

where  $M_i$  are generators of rotations  $\{M_i, M_j\} = \epsilon_{ij}^{\phantom{ij}k} M_k$ , and such that

$$\delta_\lambda \mathcal{C}(p) \equiv \lambda^i \{N_i, \mathcal{C}(p)\} = 0. \quad (44)$$

It is easy to check that such generators always exist. Notice now that since  $\mathcal{C}(p)$  is an integral of  $g^{\alpha\beta}(p) \dot{p}_\alpha \dot{p}_\beta$  with  $g^{\alpha\beta}(p) = \eta^{ab} E_a^\alpha(p) E_b^\beta(p)$ , it follows from (44) that  $E_a^\alpha(p) \dot{p}_\alpha$  transforms as a un-deformed Lorentz vector (see Sect. 5 for an explicit example), i.e.,

$$\delta_\lambda E_0^\alpha(p) = \lambda^i \{N_i, E_0^\alpha(p) \dot{p}_\alpha\} = \lambda^i E_i^\alpha(p) \dot{p}_\alpha, \quad \delta_\lambda E_j^\alpha(p) = \lambda^i \{N_i, E_j^\alpha(p) \dot{p}_\alpha\} = \lambda_j E_0^\alpha(p) \dot{p}_\alpha. \quad (45)$$

Then it follows that  $x^a$  must transform as a Lorentz vector as well  $\delta_\lambda x^0 = -x^i \lambda_i$ ,  $\delta_\lambda x^i = -x^0 \lambda^i$  and the free action is Lorentz-invariant. Notice the difference between the behavior of  $x^a$  with respect to Lorentz and translational symmetry: it transforms in a deformed way under translation, but in the standard way under Lorentz transformations.

Having checked the invariance of the free action let us turn to the interaction term. There are three possibilities.

It may happen that  $\mathcal{K}_\alpha$  just transform covariantly under the action of Lorentz symmetry defined above, i.e.,

$$\delta_\lambda \mathcal{K}_\alpha = \lambda^i \Lambda_{i\alpha}^\beta \mathcal{K}_\beta,$$

where  $\Lambda_i$  is a  $4 \times 4$  Lorentz matrix representing the boost. This is the case, for example in 2+1 gravity discussed in the preceding section (in this case the on-shell relation and  $E_a^\alpha(p) \dot{p}_\alpha$  are manifestly Lorentz scalar and vector, respectively.) Such possibility corresponds to the un-deformed Lorentz symmetry with a nontrivially deformed translational sector.

Another possibility is that  $\mathcal{K}_\alpha$  cannot be made covariant under the action of (deformed) Lorentz symmetry that leaves invariant the free action. In this case Lorentz symmetry is manifestly violated, the momentum conservation rule  $\mathcal{K}_\alpha = 0$  holds only in one specific Lorentz frame. As another side of the same coin one may consider the situation, in which  $\mathcal{K}_\alpha$  is the standard linear combination of momenta (34) but the on-shell relation (and the form of the momentum space tetrad) violates the (standard) Lorentz symmetry manifestly, which is the standard formulation of the theories with Lorentz invariant violation. The two situations described above in this paragraph are related by a change of coordinates in momentum space.

The third possibility is the most interesting as it uses the techniques of Hopf algebras.[56, 57] Consider first two particles. Their total momentum  $P_\alpha$  is given by a deformed sum of the particles' momenta

$$P_\alpha = (p \oplus q)_\alpha \quad (46)$$

We demand that this expression is Lorentz-covariant, i.e., the left hand side of eq. (46) transforms exactly like the right hand side

$$(\delta_\lambda P)_\alpha = [\delta_\lambda(p \oplus q)]_\alpha \quad (47)$$

If we assume that  $\delta_\lambda$  on the left hand side satisfies the Leibniz rule usually this condition cannot be met because

$$[\delta_\lambda(p \oplus q)]_\alpha \neq [\delta_\lambda(p) \oplus q]_\alpha + [p \oplus \delta_\lambda(q)]_\alpha .$$

However it may happen that this equation could be satisfied if we relax the Leibniz rule allowing the parameter  $\lambda$  in the second term to be replaced by momentum dependent one  $\lambda \triangleleft p$  (and allowing for the action of another symmetries belonging to the full group in question, in our case the rotations) so that<sup>7</sup>

$$[\delta_\lambda(p \oplus q)]_\alpha = [\delta_\lambda(p) \oplus q]_\alpha + [p \oplus \delta_{\lambda \triangleleft p}(q) + \dots]_\alpha , \quad (48)$$

where  $\dots$  denote other terms that might be present (see Sect. 5 for an explicit example.) Because of the close relation of the Lorentz symmetry action to the co-product structure known from the theory of Hopf algebras we say that the action (48) is characterized by a nontrivial coproduct. If (48) is satisfied the composition rule is covariant and both sides of (46) transform under Lorentz symmetry action in the same way.

There are some consistency conditions that must be satisfied in order for (48) to hold. First, assuming associativity of the composition rule<sup>8</sup> we have

$$\delta_\lambda((p \oplus q) \oplus r) = \delta_\lambda(p \oplus (q \oplus r))$$

so that

$$\delta_{\lambda \triangleleft (p \oplus q)} = \delta_{(\lambda \triangleleft p) \triangleleft q} + \dots \quad (49)$$

Moreover assuming Lorentz invariance of the identities  $(\ominus p) \oplus p = 0$ ,  $p \oplus (\ominus p) = 0$  we have

$$\delta_\lambda(\ominus p) \oplus p + (\ominus p) \oplus \delta_{\lambda \triangleleft \ominus p} p = 0 = \delta_\lambda p \oplus (\ominus p) + p \oplus \delta_{\lambda \triangleleft p}(\ominus p) . \quad (50)$$

We will discuss all this notions explicitly in Sect. 5 taking as an example the case of  $\kappa$ -Poincaré based construction.

The last symmetry of the action we will briefly discuss is the invariance with respect to the change of coordinates in momentum space. The invariance of the free action is again straightforward to see: both the on-shell relation and the components of the expression  $E_a^\alpha \dot{p}_\alpha$ ,  $a = 0, \dots, 3$  behave, by construction, as scalars under diffeomorphisms.

<sup>7</sup> This can be understood by noticing that if the relation  $\oplus$  is nontrivial, it must depend on  $p$  and  $q$  in a nontrivial way, and thus it is, in general, it cannot be inert under Lorentz transformations.

<sup>8</sup> This construction is based on the theory of Hopf algebras, in which associativity is assumed. To handle the non-associative case one would presumably have to employ Hopf algebroids.[58]

It is a nontrivial problem, however, to find out how the momentum conservation rule  $\mathcal{K}_\alpha$  transforms under the change of coordinates. To understand this let us reflect on the notion of total momentum of two particles

$$p_\alpha^{tot}(p^1, p^2) = (p^1 \oplus p^2)_\alpha.$$

The physical meaning of this expression is that there are two ways of looking at the momentum of the composite system: one can see this system as combination of two particles with momenta  $p^1$  and  $p^2$  or as a single point particle with momentum  $p^{tot}$ . Now, when we change variables  $p \mapsto P(p)$ , where  $P(p)$  is an invertible function expressing the new variables in terms of the old ones, the same duality between the total momentum and the momenta of constituents must hold. This means that the total momentum in new variables is the same function of  $p^{tot}$  as  $P$  is a function of  $p$ . Therefore

$$p^1 \oplus p^2 = p^{tot} \mapsto P^{tot}(p^{tot}) = P^{tot} \left( p^1 \oplus p^2 \Big|_{p^{1/2}=p^{1/2}(P^{1/2})} \right) \quad (51)$$

To see this rule in action consider the following change of variables in 2 dimensions  $(p_0, p) \mapsto (P_0, P)$  of the form<sup>9</sup>

$$P_0 = \kappa \sinh p_0/\kappa + \frac{p^2}{2\kappa} e^{p_0/\kappa}, \quad P = p e^{p_0/\kappa}.$$

Let the composition rule in  $p$  variables read

$$(p^1 \oplus p^2)_0 = p_0^1 + p_0^2, \quad (p^1 \oplus p^2) = p^1 + e^{-p_0^1/\kappa} p^2.$$

Therefore

$$\begin{aligned} P^{tot} &= \left( p^1(P^1) + e^{-p_0^1(P^1)/\kappa} p^2(P^2) \right) e^{(p_0^1(P^1) + p_0^2(P^2))/\kappa} \\ &= P^1 e^{p_0^2/\kappa} (P^2) + P^2 = (P^1 \oplus P^2)_1. \end{aligned}$$

Similar expression can be derived for the zeroth component of the total momentum. It should be noticed that the above definition of how the composition of momenta transforms under diffeomorphisms agrees with the rule of transformation of coproduct of a Hopf algebra under change of basis, as it should. It is worth noticing that when one uses the expansion near the origin of the momentum space  $\mathcal{O}$

$$(p^1 \oplus p^2)_\alpha = p_\alpha^1 + p_\alpha^2 + \Gamma_\alpha^{\beta\gamma} p_\beta^1 p_\gamma^2$$

and makes a change of coordinates according to recipe described above, one finds that  $\Gamma_\alpha^{\beta\gamma}$  transforms as a connection, which justifies dubbing  $\Gamma_\alpha^{\beta\gamma}$  the connection coefficients[22, 51]).

Similar definition holds for the operation  $\ominus$  and using them one can reconstruct the form of the conservation rule for an arbitrary vertex after change of momentum space variables.

This concludes our discussion of the particle action and its symmetries.

## SPACETIME AND RELATIVE LOCALITY

In the previous sections we concentrated our investigations on curved momentum spaces. Let us now investigate what is the fate of spacetime in theories, in which the momentum space is nontrivial. As we will see below it turns out to be the major difference between the spacetimes of special (and general) relativity and the present case is the necessity to replace the absolute notion of locality valid for all observers with the somehow relaxed notion of relative locality.

It should be stressed at the very beginning that in order to formulate the free action (33) or (36) we must have in our disposal the spacetime structure, with coordinates  $x^a$  defined in such a way that the worldlines of free particles are straight lines. This is essentially guaranteed in special relativity because, operationally, to define coordinates one uses freely propagating light signals<sup>10</sup> and the standard coordinates are adjusted so as to make the worldlines straight.

<sup>9</sup> This example illustrates the transition from the bicrossproduct to the classical basis in  $\kappa$ -Poincaré; see Sect. 5 for more details.

<sup>10</sup> In principle one is free to use some other freely moving probes, but for the sake of concreteness we will discuss only light signals here.

In the curved momentum space case (36) the situation is more complex, because the coordinatizations done with the help of light signals of different energies will lead, in general, to different resulting coordinates. One can circumvent this problems by using light signals with an appropriately low energy, so as to make the impact of a nontrivial momentum space geometry negligible. In classical theory, where we think of the light signal in terms of a bunch of pointlike massless particles, this is, of course, not problematic. In quantum theory, however, such procedure is questionable because quantum mechanically we have to do with waves, not point particles, and therefore the resolution of the low-energetic probes is bounded from below by their wavelength. This causes a fundamental problem, but for all practical purposes it is not really relevant, because if the scale  $\kappa$  is of order of Planck energy, one can achieve, for example, the atom size resolution by using photons with  $p_0/\kappa \sim 10^{-25}$ , for whose the effects resulting from curved momentum space are completely negligible.

Having defined the spacetime coordinates with the help of low energy light signals we turn now to the question of how this spacetime is related to the ones corresponding to particles carrying momenta of the magnitude comparable to the scale  $\kappa$ . Geometrically speaking these spacetimes are spaces cotangent to the momentum manifold at some appropriate point  $\mathcal{P}$  with coordinates  $p_\alpha$ . Such spaces are isomorphic as vector spaces but there is, in principle, no canonical isomorphism relating the spacetimes at different points  $\mathcal{P}$  and  $\mathcal{Q}$  if  $\mathcal{P} \neq \mathcal{Q}$ . It should be noticed however that this same problem appears already in special relativity and is solved there by assuming that the flat momentum space geometry possesses translational Killing vectors which make it possible to translate rigidly all the spacetimes to one point, the origin  $\mathcal{O}$  of the momentum space, say.

On the other hand, in general relativity one solves the analogous problem of comparing tangent spaces at the different spacetime points by using the tetrad  $e_\mu^a(x)$  that maps all these spaces into one ‘ambient’ Minkowski space:  $T_x M \ni v^\mu(x) \mapsto e_\mu^a(x) v^\mu(x) = v^a \in Mink_4$ .

We made use of the same trick in constructing our free action (36). There was a momentum space tetrad there that effectively made the coordinates  $x^a$  belong to a single Minkowski spacetime, independently of the point in the momentum space. Moreover, since the curved momentum space effect are to vanish at the momentum space origin  $\mathcal{O}$  (by correspondence principle which guarantees that at appropriately small momenta our theory becomes identical with special relativity) this Minkowski spacetime is just the ones we coordinatized with the help of the low energy light signals, as described above. Having constructed the positions all belonging to the same spacetime we are now free to compare positions of wordlines of particles carrying different momenta. Moreover, since as we have shown in Sect. 3, if the Lorentz symmetry is present the coordinates  $x^a$  transform as Lorentz vectors, one can equip the spacetime with the standard Minkowski metric, which makes it possible to compute distances between points in the standard way.

There is a prize to be paid, however. The kinetic term in (36) is non-trivial, which leads to the ‘non-commutativity’ of spacetime coordinates, expressed in the classical theory by the fact that their Poisson bracket is not equal to zero. This can be seen directly by calculating the symplectic form and then the Poisson brackets, but here we will use a simpler way to compute them.

Instead of the  $x^a$  variables, being coordinates in the universal spacetime we discussed above (36), let us introduce the cotangent space ones by

$$x^\alpha(p) \equiv x^a E_a^\alpha(p).$$

In terms of  $x^\alpha$  the kinetic term is  $-x^\alpha \dot{p}_\alpha$  so that the only non-vanishing Poisson bracket reads

$$\{x^\alpha, p_\beta\} = \delta_\beta^\alpha.$$

Returning to the physical coordinates using the inverse tetrad  $x^a = E_\alpha^a(p) x^\alpha$  we find

$$\{x^a, p_\beta\} = E_\beta^a(p) \tag{52}$$

$$\{x^a, x^b\} = E_\alpha^a E_\beta^b (E_c^{\alpha,\beta} - E_c^{\beta,\alpha}) x^c, \tag{53}$$

with momenta having vanishing Poisson bracket, so that indeed in the case of any non-trivial momentum space geometry the position variables have a non-vanishing Poisson bracket. This is a nice illustration of the idea that curved momentum spaces are in one-to-one relation with non-commutative geometry[59].

Having discussed the spacetime aspect of the free action, let us now turn to the interaction part. Here we encounter the notion of *relative locality*. Let us take a vertex with some wordlines coming in and some going out, which is characterized by the interaction coordinates  $z^a$ . The equation following equation (41) relates the coordinate of the end of the wordline of a particle carrying momentum  $p$  with the interaction one

$$x^a(0) = E_\beta^a(p) z^\alpha \frac{\partial \mathcal{K}_\alpha(p, p^{(1)}, \dots)}{\partial p_\beta}. \tag{54}$$

The first thing which should be noticed is that if the momenta of all the particles are very small compared to the scale  $\kappa$ ,  $\mathcal{K}_\alpha(p, p^{(1)}, \dots) \sim p_\alpha + p_\alpha^{(1)} + \dots$ ,  $E_\beta^a(p) \sim \delta_\beta^a$  and the worldline and interaction coordinates coincide. Moreover if the interaction event takes place at the origin of a coordinate system, of Alice, say, so that  $z^\alpha = 0$  it follows from (54) that  $x^a(0) = 0$  as well, for arbitrary interaction and the magnitude of momenta.

This changes however when the interaction event takes place away from the origin of the Alice's coordinates. Then  $x^a(0) \neq z^\alpha$ . But the spacetime point with coordinates  $z^\alpha$  is an origin of some another, translated coordinate system, say of Bob, and by the same equation (54), for Bob  $x^a(0) = z^\alpha = 0$ . This shows that in the theories with curved momentum space the concept of locality loses its absolute status and becomes a notion relative to the observer.

This may sound as a very radical departure from the standard physical intuitions, but it should be noted that operationally speaking all the physical experiments are performed at the origin of the observer's coordinate system, and the knowledge of distant events is only inferred with the help of some (light or otherwise) signals sent from the events to the observer. What really matters is whether the relativity principle holds, i.e., whether the pictures of a physical process inferred by different observers are mutually consistent, so that neither of them can be treated as a privileged one.

The relative feature of locality has been nicely illustrated in the recent paper[60], in the framework of  $\kappa$ -Poincaré. Consider an observer Alice who sends two light signals of different energies (one low-energetic and second of very high energy) in the direction of a distant observer Bob, who is static with respect to Alice. According to Alice these two signals reach Bob at exactly the same spacetime point, because they were sent simultaneously from her location and, in  $\kappa$ -Poincaré the velocity of massless particles is energy-independent (see next Section.) On the other hand Bob does not see these two particles arriving simultaneously, for him there is a time lag between arrival time of order of  $D \Delta E / \kappa$ , where  $D$  is the distance between Alice and Bob and  $\Delta E$  the difference of particles' energies. How this can be reconciled with the fact that for Bob too all massless particles move with the universal speed of light? The answer turns out to be simple: for Bob the events of particles' creation at Alice's location are not simultaneous, but there is a time lag between them, exactly the same as the time lag of the particles' arrival time. Thus even if Alice does not agree with Bob about which events are local and which are not, their points of view are perfectly mutually consistent, in a relativistic way.

A similar analysis can be made in the case of quantum mechanics[61]. Here we also have two distant observers, Alice and Bob, at relative rest, who create two identical Gaussian wave packets at their origins. In Alice's description the packet at her origin is perfectly spherically symmetric, but the one at Bob's has larger 'fuzziness.' According to relative locality, Bob's view should be exactly the same, just with wave packets changing roles. The direct computation presented in[61] confirms this expectation.

Let us finish this section asking a question if relative locality is a real physical effect or perhaps just a (spacetime) coordinate artifact? As noted above even special relativity can be artificially made a theory with relative locality, by replacing the standard Minkowski space coordinates with some new, momentum dependent ones (although in order to do so one has to introduce a mass scale, which is not really available in special relativity.) Then the question arises if in the theories with curved momentum space the relative locality property can be undone by changing position variables? There is no general answer to this question; it can be checked however that in the case of  $\kappa$ -Poincaré construction, to be discussed in more details in the next section, when the nontrivial structure of momentum space and deformation of spacetime symmetries are both 'genuine', i.e., cannot be undone by any change of variables, relative locality is genuine too.

This concludes our brief presentation of the role of spacetime in the theories with curved momentum space and the principle of relative locality.

## $\kappa$ -POINCARÉ

Till now we we discussed the particles' model in a rather abstract way, without referring to any particular example. Let us therefore turn now to a specific model, which has its roots in  $\kappa$ -Poincaré algebra and  $\kappa$ -Minkowski space constructions[15–18]. In what follows we will borrow from the presentations in Refs. [62] and [57].

Let us start with describing the momentum space of a  $\kappa$ -Poincaré particle. It is a four dimensional group manifold of a Lie group  $AN(3)$ , whose Lie algebra generators satisfy

$$[X^0, X^i] = \frac{i}{\kappa} X^i. \quad (55)$$

Sometimes  $X^a$  above are interpreted as positions in a non-commutative spacetime; such spacetime is known under the name of ' $\kappa$ -Minkowski space'[17]. For our purposes the relevant matrix representation of this Lie algebra will happen

to be the 5-dimensional one, in which case we have

$$X^0 = -\frac{i}{\kappa} \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix} \quad \mathbf{X} = \frac{i}{\kappa} \begin{pmatrix} 0 & T & 0 \\ \epsilon & \mathbf{0} & \epsilon \\ 0 & -\epsilon^T & 0 \end{pmatrix}, \quad (56)$$

where bold fonts are used to denote space components of a 4-vector and  $\epsilon$  is a three dimensional vector with a single unit entry, e.g.,  $\epsilon^1 = (1, 0, 0)$ . This algebra has one abelian generator  $X^0$  and three nilpotent generators  $X^i$  (one can check that  $(X^i)^3 = 0$ ) which is the reason behind the notation  $AN(3)$ <sup>11</sup>. Let us now consider a group element of  $AN(3)$  (sometimes called the ‘ordered plane wave on  $\kappa$ -Minkowski space’[64])

$$u(p) = e^{ip_i X^i} e^{ip_0 X^0}. \quad (57)$$

In the representation (56) this is a  $5 \times 5$  matrix which acts on 5-dimensional Minkowski space as a linear transformation. One finds

$$\exp(ip_0 X^0) = \begin{pmatrix} \cosh \frac{p_0}{\kappa} & \mathbf{0} & \sinh \frac{p_0}{\kappa} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \sinh \frac{p_0}{\kappa} & \mathbf{0} & \cosh \frac{p_0}{\kappa} \end{pmatrix}, \quad \exp(ip_i X^i) = \begin{pmatrix} 1 + \frac{\mathbf{p}^2}{2\kappa^2} & \frac{\mathbf{p}}{\kappa} & \frac{\mathbf{p}^2}{2\kappa^2} \\ \frac{\mathbf{p}}{\kappa} & \mathbf{1} & \frac{\mathbf{p}}{\kappa} \\ -\frac{\mathbf{p}^2}{2\kappa^2} & -\frac{\mathbf{p}}{\kappa} & 1 - \frac{\mathbf{p}^2}{2\kappa^2} \end{pmatrix},$$

where  $\mathbf{1}$  is the unit  $3 \times 3$  matrix from which we find

$$u(p) = \begin{pmatrix} \frac{\bar{P}_4}{\kappa} & e^{-p_0/\kappa} \frac{\mathbf{P}}{\kappa} & \frac{P_0}{\kappa} \\ \frac{\mathbf{P}}{\kappa} & \mathbf{1} & \frac{\mathbf{P}}{\kappa} \\ \frac{\bar{P}_0}{\kappa} & -e^{-p_0/\kappa} \frac{\mathbf{P}}{\kappa} & \frac{P_4}{\kappa} \end{pmatrix}. \quad (58)$$

We now take a special point in this space, the origin  $\mathcal{O}$  with coordinates  $(0, \dots, 0, \kappa)$ <sup>12</sup> and act with (58) on it. As a result we get a point in the 5-dimensional Minkowski space with coordinates  $(P_0, P_i, P_4)$  with

$$\begin{aligned} P_0(p_0, \mathbf{p}) &= \kappa \sinh \frac{p_0}{\kappa} + \frac{\mathbf{p}^2}{2\kappa} e^{p_0/\kappa}, \\ P_i(p_0, \mathbf{p}) &= p_i e^{p_0/\kappa}, \\ P_4(p_0, \mathbf{p}) &= \kappa \cosh \frac{p_0}{\kappa} - \frac{\mathbf{p}^2}{2\kappa} e^{p_0/\kappa}. \end{aligned} \quad (59)$$

It is easy to check that

$$-P_0^2 + \mathbf{P}^2 + P_4^2 = \kappa^2 \quad (60)$$

and therefore the group  $AN(3)$  is isomorphic, as a manifold to 4-dimensional de Sitter space. Actually, this group is not the whole of this space, but rather a half of it because it follows from (59) that

$$P_0 + P_4 = e^{p_0/\kappa} > 0, \quad P_4 \equiv \sqrt{\kappa^2 + P_0^2 + \mathbf{P}^2} > 0. \quad (61)$$

To construct the free action with the group  $AN(3)$  as a momentum space, let us recall that in this case there exists a canonical form of the kinetic term provided by the Kirillov symplectic form[65]. It can be constructed as

<sup>11</sup> In the  $\kappa$ -Poincaré literature this algebra is usually called the  $\kappa$ -Minkowski algebra and is defined as a dual (in the Hopf algebraic sense) to the  $\kappa$ -Poincaré Hopf algebra, see Refs. [17, 18] and [63] for more details and additional references.

<sup>12</sup> Had we chosen another point as  $\mathcal{O}$ , we would get different realizations of the group  $AN(3)$ . For example if  $\mathcal{O} = (\kappa, \dots, 0)$  we would get the Euclidean realization and for  $\mathcal{O} = (\kappa, 0, 0, 0, \kappa)$  the light-cone one.



follows. Since the positions belong to the cotangent space of the momentum space they can be naturally associated with elements of a dual to the Lie algebra, which is a linear space spanned by the basis  $\sigma_a$  with the pairing

$$\langle \sigma_a, X^b \rangle = \frac{1}{i} \delta_a^b.$$

Now having a group element  $\mathbf{u}(p)$  (57) we know that  $\mathbf{u}^{-1}\dot{\mathbf{u}}$  is an element of the Lie algebra  $AN(3)$  and thus we can define the kinetic term of the action as<sup>13</sup>

$$S^{kin} = - \int d\tau x^a \langle \sigma_a, \dot{\mathbf{u}} \mathbf{u}^{-1} \rangle = - \int d\tau x^0 \dot{p}_0 + e^{p_0/\kappa} x^i \dot{p}_i, \quad (62)$$

where the last equality is obtained by substituting the explicit form of  $\mathbf{u}(p)$ , (57). From (62) we can immediately read of the components of the momentum space tetrad

$$E_0^0 = 1, \quad E_i^j = e^{p_0/\kappa} \delta_i^j, \quad (63)$$

and the line element takes the form

$$ds^2 = -dp_0^2 + e^{2p_0/\kappa} d\mathbf{p}^2, \quad (64)$$

where  $d\mathbf{p}^2 \equiv dp_i dp_i$ , which is nothing but the metric of de Sitter space in ‘flat’ coordinates. It should be noticed that the metric (64) can be obtained as an induced metric  $ds^2 = -dP_0^2 + d\mathbf{P}^2 + dP_4^2$  when (59) is used.

Having the metric we can calculate the distance function, which is used to define the mass shell condition; it reads[57, 66]

$$\mathcal{C}(p) = \kappa^2 \operatorname{arccosh} \frac{P_4}{\kappa}$$

so that the mass-shell condition reads

$$\cosh \frac{p_0}{\kappa} - \frac{\mathbf{p}^2}{2\kappa^2} e^{p_0/\kappa} = \cosh \frac{m}{\kappa}. \quad (65)$$

Using (62) and (65) we construct the free particle action

$$S_{free}^{\kappa P} = - \int d\tau \left( x^0 \dot{p}_0 + e^{p_0/\kappa} x^i \dot{p}_i + 2N \left[ \kappa^2 \cosh \frac{p_0}{\kappa} - \frac{\mathbf{p}^2}{2} e^{p_0/\kappa} - \kappa^2 \cosh \frac{m}{\kappa} \right] \right), \quad (66)$$

where we replaced  $N$  with  $2N$  so that the action  $S_{free}^{\kappa P}$  reduces to the standard relativistic particle action in the limit  $\kappa \rightarrow \infty$ . It is worth noticing that the action (66) leads to a nontrivial Poisson brackets algebra

$$\begin{aligned} \{x^0, x^i\} &= -\frac{1}{\kappa} x^i, & \{x^i, x^j\} &= 0 \\ \{x^0, p_0\} &= 1, & \{x^i, p_j\} &= e^{-p_0/\kappa} \delta_j^i, & \{x^0, p_i\} &= \{x^i, p_0\} = 0, \end{aligned} \quad (67)$$

with  $p_\alpha$  having vanishing Poisson brackets.

The equations of motion following from (66) have the form

$$\begin{aligned} \dot{p}_\alpha &= 0 \\ \dot{x}^i &= -2N p_i, \\ \dot{x}^0 &= 2N \left( \kappa \sinh \frac{p_0}{\kappa} - \frac{\mathbf{p}^2}{2\kappa} e^{p_0/\kappa} \right), \end{aligned} \quad (68)$$

---

<sup>13</sup> Had we chosen to use  $\dot{\mathbf{u}} \mathbf{u}^{-1}$  instead we would get an action  $S^{kin} = - \int d\tau x^\alpha \dot{p}_\alpha - (\kappa)^{-1} x^i p_i \dot{p}_0$ , which is related to (62) by the change of variables,  $\mathbf{p} \mapsto \tilde{\mathbf{p}} = \mathbf{p} e^{p_0/\kappa}$ .

supplemented by the mass shell condition (65), where in the equations for  $x^a$  we omitted terms proportional to  $\tau$  derivatives of momenta. It is worth noticing that the velocity of massless particles

$$\mathbf{v}^2 = \left| \frac{\dot{x}^i}{\dot{x}^0} \right|^2 = 1$$

so that the velocity of light is independent of the energy of photons[67].

Having discussed the free part of the  $\kappa$ -Poincaré particle action let us now turn to the interactions. In order to construct the interaction term in the action, we must define how the momenta are composed and what is the form of the inverse (antipode). They can be directly inferred from the group composition and group inverse, respectively.

Having two group elements  $u(p)$  and  $u(q)$  (57) we can multiply them forming another group element

$$u(p)u(q) \equiv u(p \oplus q) \quad (69)$$

with

$$(p \oplus q)_0 = p_0 + q_0, \quad (p \oplus q)_i = p_i + e^{-p_0/\kappa} q_i. \quad (70)$$

Similarly we define the antipode

$$u^{-1}(p) \equiv u(\ominus p) \quad (71)$$

so that

$$(\ominus p)_0 = -p_0, \quad (\ominus p)_i = -e^{p_0/\kappa} p_i. \quad (72)$$

Clearly,  $p \oplus (\ominus p) = (\ominus p) \oplus p = 0$ . Using these two operations and remembering that  $\oplus$  is associative, we can construct  $\mathcal{K}_\alpha$  for an arbitrary vertex.

Thus the  $\kappa$ -Poincaré particle action has the form

$$S = \sum_{\mathcal{I}} S_{free}^{\kappa P} + \sum_{\mathcal{V}} z^\alpha \mathcal{K}_\alpha, \quad (73)$$

where in the first term we have a sum of free  $\kappa$ -Poincaré particle actions (66) with an appropriate choice of the integration ranges, and the second term is a sum of vertices constructed with the help of  $\oplus$  and  $\ominus$  operations (70), (72). This concludes the construction of the  $\kappa$ -Poincaré particles' dynamics. Let us now turn to the symmetries of the  $\kappa$ -Poincaré particles' system.

### $\kappa$ -Poincaré symmetry

The free  $\kappa$ -Poincaré particle actions (66) is invariant under both translational and Lorentz symmetries.

The explicit form of translational transformations that leave the free action invariant in the absence of interactions can be easily deduced from the general discussion presented in Sect. 3.1 and reads

$$\delta x^0 = \xi^0, \quad \delta x^i = \xi^i e^{-p_0/\kappa}. \quad (74)$$

It follows that the conserved Noether charges associated with the translational symmetry are the components of the momentum, as usual.

In the case of a single vertex, the transformation rule can be read off from the general formula (42) and read

$$\delta x_{\mathcal{I}}^0 = \xi^\gamma \frac{\partial \mathcal{K}_\gamma}{\partial p_{\mathcal{I}}^0}, \quad \delta x_{\mathcal{I}}^i = \xi^\gamma \frac{\partial \mathcal{K}_\gamma}{\partial p_{\mathcal{I}}^i} e^{-p_0/\kappa}, \quad \delta z^\alpha = \xi^\alpha. \quad (75)$$

The case of more than one vertex is much more complicated and is discussed in details in[66].

Let us now turn to Lorentz symmetry. The Lorentz transformations of momenta  $p_\alpha$  can be easily found if one recalls that the components  $P_\alpha$  in (59) transform as components of a Lorentz vector, while  $P_4$  is a Lorentz scalar. This follows from the fact that the four dimensional Lorentz group acts on four dimension subspace of the five

dimensional Minkowski space, in which the  $AN(3)$  group manifold is imbedded in the standard linear way and leaves the fifth dimension invariant. Thus we have

$$\delta_\lambda P_0 = \lambda^i P_i, \quad \delta_\lambda P_i = \lambda_i P_0, \quad \delta_\lambda P_4 = 0.$$

From these equations we deduce that  $P_4$  is proportional to the Casimir used in the mass-shell relation, which we indeed derived above (65). Further, using the chain rule we can calculate from them the transformation laws of  $p_\alpha$ , which turn out to be

$$\delta_\lambda p_0 = p_i, \quad \delta_\lambda p_i = \lambda_i \left( \frac{\kappa}{2} (1 - e^{-2p_0/\kappa}) + \frac{\mathbf{p}^2}{2\kappa} \right) - \frac{1}{\kappa} \lambda^j p_j p_i. \quad (76)$$

Let us return to the line element (64). One can check by direct calculation that it is Lorentz invariant<sup>14</sup>

$$\delta_\lambda ds^2 = 0, \quad (77)$$

and it follows that

$$\delta_\lambda E_0^\alpha dp_\alpha = \lambda^i E_i^\alpha dp_\alpha, \quad \delta_\lambda E_i^\alpha dp_\alpha = \lambda_i E_0^\alpha dp_\alpha,$$

from which we deduce that the positions transform as components of a Lorentz vector

$$\delta_\lambda x^0 = -\lambda_i x^i, \quad \delta_\lambda x^i = -\lambda^i x^0 \quad (78)$$

in agreement with the general result presented in Sect. 3.

Having discussed the action of Lorentz transformations in the case of the single particle free action, let us now turn to the interaction term. We will discuss only the composition law for two particles and the antipode; these results can be easily generalized to the case of many particles.

Consider the composition law first. In order to secure the Lorentz invariance it must be covariant i.e., in the equality

$$p_\alpha^{tot} = (p \oplus q)_\alpha$$

both sides must transform in exactly the same way. As we will see this condition cannot be met if one insists on the Leibnizian action of the symmetry transformations on the right hand side. Let us recall the component form the composition law (70)

$$p_0^{tot} = p_0 + q_0, \quad p_i^{tot} = p_i + e^{-p_0/\kappa} q_i$$

It is clear that both sides of these equations are covariant with respect to rotations: both sides of the equations are rotation scalars in the first and components of a three-vector in the second.

Consider first the equation for the zeroth component of momentum. The left hand side transforms as

$$\delta_\lambda p_0^{tot} = \lambda^i (p_i + e^{-p_0/\kappa} q_i)$$

Following the general idea presented in Sect. 3 for the right hand side we take

$$\delta_\lambda (p_0 + q_0) = \delta_\lambda p_0 + \delta_{\lambda \triangleleft p} q_0$$

and we deduce that

$$\delta_{\lambda \triangleleft p} = \delta_{\bar{\lambda}} + \dots, \quad \bar{\lambda}_i = e^{-p_0/\kappa} \lambda_i$$

where  $\dots$  denote possible terms that vanish when acting on  $q_0$ . Such terms might be only proportional to rotations  $\delta_\rho$ . Checking the covariance of the space part of the composition law one finds that

$$\delta_{\lambda \triangleleft p} = \delta_{\bar{\lambda}} + \delta_\rho, \quad \rho_i = \epsilon_i^{jk} \lambda_j p_k. \quad (79)$$

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<sup>14</sup> This follows immediately from the manifest Lorentz invariance of  $ds^2 = -dP_0^2 + d\mathbf{P}^2 + dP_4^2$ .

It remains to be checked what are the covariance properties of the antipode  $\ominus$ . The covariance means that taking Lorentz transformation first and the antipode of the result must be equal to the action of a modified Lorentz transformation with the parameter  $\ominus\lambda$  on the antipode, i.e.,

$$\delta_{\ominus\lambda} \ominus p = \ominus(\delta_\lambda p).$$

Since the antipode (72) is manifestly rotationally covariant, we can just concentrate on Lorentz boosts, as before. Then

$$\delta_{\ominus\lambda}(\ominus p)_0 = -(\ominus\lambda)^i p_i = \ominus(\lambda^i p_i) = -e^{p_0/\kappa} \lambda^i p_i,$$

from which it follows that

$$\ominus\lambda_i = e^{p_0/\kappa} \lambda_i + \dots,$$

where  $\dots$  denotes, as before, a possible contribution from rotation. From the covariance of the spacial components of the antipode one can identify this rotational transformation and the final result is

$$\delta_{\ominus\lambda} = e^{p_0/\kappa} (\delta_\lambda - \delta_\rho), \quad \rho_i = \epsilon_i^{jk} \lambda_j p_k. \quad (80)$$

It should be noticed that the derived action of Lorentz transformations on the composition of momenta and the antipode is in one-to-one correspondence with the co-product and the antipode of Lorentz generators of the  $\kappa$ -Poincaré Hopf algebra[15–18].

Having discussed the infinitesimal action of Lorentz transformations let us finally turn to the apparent problem that seemingly plagues the theory when we allow for the finite Lorentz transformations. It was noticed already in the early paper[9] and discussed further in[56, 57]. The problem is that although the Lorentz transformations are well defined in the case of positive anergy  $p_0 > 0$  for all real values of the boost parameter, for negative energy  $p_0 < 0$  this is not the case and only some finite interval of boost parameters is allowed. An easy way to see this is to realize that the condition (61) defining the momentum space is clearly *not* Lorentz invariant. It is worth stressing here that this problem is not really important in the classical theory of particles, where the energies are always positive. In the case of a field theory with  $AN(3)$  momentum space the problem seems to be indeed severe, and, if there, it would mean that the Lorentz invariant field theory with this kind of a momentum space may not exist at all (see Refs. [62] and [68] for details.)

The solution of this apparent problem has been proposed in Ref. [69]. The idea is to modify the way the Lorentz transformation act on positive and negative energies with the antipode action (with the parameter  $\ominus\lambda$ ) employed in the latter case. It turns out that such action is fully consistent and free of problems mentioned above.

## CONCLUSIONS

In this paper we reviewed motivations leading to and some properties of the relativistic theory of particles with curved momentum space. Let us conclude this presentation with the list of of the most pressing problems that need to be solved before this theory reaches a mature stage.

Although it is very encouraging that models with curved momentum space can be rigorously derived from the theory of gravity coupled to point particles in 2+1 spacetime dimensions one has to be able to derive models with curved momentum space also in the physical 3+1 spacetime dimensions. As discussed in Sect. 2 there are indications that such model do exist, but their explicit construction will certainly put the whole framework on a much firmer ground.

In absence of such explicit construction it is quite important to examine the internal consistency of models with curved momentum space and their symmetries. As we mentioned in the main text already in the case of tree interactions with many vertices it is quite hard to prove the invariance of the model under global spacetime translations. When the loop processes<sup>15</sup> are allowed, the preliminary investigations suggest that the spacetime translational symmetry is lost, and, even worse, the very notion of causality might be at stake[70]. It is not clear how these results, if confirmed, are to be interpreted.

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<sup>15</sup> As an example of a loop process consider some number of particles interacting in vertex  $A$  in such a way that two, or more, of the particles interacting at  $A$  interact again in another vertex  $B$ .

Also would be very interesting to investigate properties of classical and especially quantum field theories in the case of curved momentum space. The case of momentum space being a group manifold is especially interesting, because in this case the symmetries have a mathematical structure of a Hopf algebra, being an appropriate deformation of the Poincaré algebra. If a consistent interacting quantum field theory can be constructed, either in 3+1 or in 2+1 dimensions, it would serve as an explicit counterexample of the celebrated Coleman-Mandula theorem.

Last but not least, as discussed above, the theories with curved momentum space enjoy the relative locality properties, which leads to the prediction that the time of flight of light coming from distant sources may become energy-dependent, with the time lag proportional to the energy difference and distance. As we discussed it is not clear that for photons the proportionality factor should be of order of the inverse Planck energy, but if it is such effects might be within reach of current experiments[71, 72].

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